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ON THE STRESS TENSOR IN A FLUID CONTAINING DISPERSE PARTICLES

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An incompressible fluid containing a large number of spherical particles per unit volume, is considered. The volume concentration of these particles $c \ll 1$, and the volume of the particles themselves can change its value.

Let the motion of the fluid be potential. Then the velocity of the particles $\mathbf{u}$ and the mass averaged velocity of the fluid $\mathbf{v}$ can both be found using the procedure described in [1] for the case of bubbles. In solving the dynamic problem we neglect the terms $c\mathbf{w} = c(\mathbf{u} - \mathbf{v})$ from the equations and determine the quantity $\mathbf{v}$, with the terms $c\mathbf{w}$ taken into account, from the kinematic relations formulated in [1, 2]. In connection with problems in which the flows are not potential, it is expedient to obtain the impression for the stress tensor of the medium in the potential approximation, in order to assess phenomenologically the inertial effects in the absence of potential.

Let us consider the Cauchy-Lagrange integral averaged over the fluid volume

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\nabla \Phi|^2 + \frac{p}{\rho} - U = f(t)$$

We can find the mean value of $d\Phi / dt$ using the procedures of differentiation described in [1]

$$\left\langle \frac{\partial \Phi}{\partial t} \right\rangle = \frac{\partial}{\partial t} \langle \Phi \rangle - \frac{1}{2} c\mathbf{u}\mathbf{w} \tag{1}$$

Averaging $|\nabla \Phi|^2$ to within c^2, it is sufficient to write

$$|\nabla \Phi|^2 \approx |\nabla \langle \Phi \rangle|^2 + 2\nabla \langle \Phi \rangle \nabla (\Phi - \langle \Phi \rangle) \tag{2}$$

Taking into account the expression for the mean-mass velocity

$$\mathbf{v} = \langle \nabla \Phi \rangle = \nabla \langle \Phi \rangle + {}^1/_2\, c\mathbf{w} \tag{3}$$

obtained in [1, 2] we find, using (1) and (2), the mean value of the Cauchy-Lagrange integral

$$\frac{\partial \langle \Phi \rangle}{\partial t} + \frac{1}{2} |\nabla \langle \Phi \rangle|^2 - \frac{1}{2} cw^2 + \frac{\langle p \rangle}{\rho} - U = f(t) \tag{4}$$

Differentiating (4) with respect to the coordinate and taking (3) into account, we can write

$$\frac{\partial}{\partial t}\left(\mathbf{v} - \frac{c\mathbf{w}}{2}\right) + \left(\left(\mathbf{v} - \frac{c\mathbf{w}}{2}\right)\nabla\right)\left(\mathbf{v} - \frac{c\mathbf{w}}{2}\right) = \nabla U - \frac{1}{\rho} \nabla \left(\langle p \rangle - \frac{\rho}{2} cw^2\right) \tag{5}$$

The above expression represents, in fact, the equation of motion of the fluid phase. To obtain the expression for the stress tensor, it is sufficient to reduce the above expression to the canonical form of the equations of motion of a continuum. This requires simple transformations to within the terms of order of the square of concentration, with the equation of continuity for the disperse phase

$$\frac{\partial c}{\partial t} + \operatorname{div} cu = 3c \frac{R}{R}$$

and the equation of motion of particles in an inhomogeneous flow [3]

$$\frac{4}{3}\pi R^3\rho_b\frac{d\mathbf{u}}{dt}=\frac{4}{3}\pi R^3\rho_b\mathbf{g}+2\pi R^3\rho\left(\frac{d\mathbf{v}}{dt}-\frac{1}{3}\frac{d\mathbf{u}}{dt}-\frac{R^{\cdot}}{R}\mathbf{w}-\frac{2}{3}\mathbf{g}\right)$$

both taken into account. Here R is the radius of the particles, ρ_b their mean density and $\mathbf{g}=\nabla U$. The differentiation of $\mathbf{v}$ is carried out along the trajectory of a fluid particle and that of $\mathbf{u}$, along the trajectory of the center of the particle. Finally, Eq. (5) assumes the form

$$\rho(1-c)\frac{d\mathbf{v}}{dt}=\rho(1-c)\mathbf{g}+\mathbf{F}+(\nabla\mathbf{P}) \tag{6}$$

$$P_{ij}=\left(-\langle p\rangle+\frac{1}{2}\rho cw^2\right)\delta_{ij}-\frac{1}{2}\rho cw_iw_j,\quad \mathbf{F}=\rho_bc\left(\mathbf{g}-\frac{d\mathbf{u}}{dt}\right)$$

The quantity $\mathbf{F}$ represents the forces acting on the fluid from the disperse particles, and $\mathbf{P}$ is the stress tensor. The pressure in the fluid is equal to $-{}^1/_3\ P_{ii}=\langle p\rangle-{}^1/_3\rho cw^2$. The author of [4] showed the existence of the terms in P_{ij} proportional to cw_iw_j, however the exptession for the tensor P obtained by him was different. The last term in the expression (6) for P_{ij} was obtained earlier by R. I. Nigmatulin with help of the cellular model.

The influence of the viscous forces can be accounted for in an approximate manner by including them in the forces of phase interaction.

The author of [2] derived the equations of motion of the bubbles under erroneous assumptions. His equations (4.2) and (4.3) are valid for a single bubble but fail, within the limits of accuracy used, for a multibubble medium. Indeed, averaging the discrete function $\partial\Phi_\alpha'/\partial t$ we obtain, according to [1],

$$\overline{\frac{\partial\Phi_\alpha'}{\partial t}}=\frac{\partial}{\partial t}\overline{\Phi_\alpha'}-\frac{1}{2}cu_i(w_i+2A_{ij}w_j) \tag{7}$$

and using the formulas (2.9) and (2.21) of [1] we average the equation for the change of the bubbles radii as follows:

$$RR^{\cdot\cdot}+\frac{3}{2}R^{\cdot 2}-\frac{\partial\langle\Phi\rangle}{\partial t}-\frac{1}{2}|\nabla\langle\Phi\rangle|^2-\frac{1}{4}(\mathbf{u}-\nabla\langle\Phi\rangle)^2+U+f- \tag{8}$$

$$\frac{\bar{p}_g}{\rho}+\frac{2\sigma}{\rho R}+\frac{3}{4}cw^2+\frac{3}{2}cA_{ij}w_iw_j=0$$

Here the tensor A_{ij} depends on the microstructure of the medium [1]. The last term of (7) is not taken into account in [2]. Two last terms of (8) are absent from the equation (4.3) of [2], nor do the analogous terms appear in (4.2). Thus, the dynamic equations of [2] cannot be used to derive the expression for the stress tensor of the fluid phase. In contrast with [2], the authors of [1] derived the equations of motion of the bubbles neglecting the terms of the order of $c\mathbf{w}$ in all equations including the kinematic ones, since it would be incorrect to leave them in. Terms of the order of $c\mathbf{w}$ are taken into account only when computing the motion of the fluid phase. To include the terms of the order of $c\mathbf{w}$ and cw_iw_j in the equations of motion of the bubbles would necessitate the determination of the tensor A_{ij} depending on the relative distribution of the bubbles with respect to each other, and this can be obtained using e.g. methods of the kinetic theory.

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